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Homological Aspects of Noetherian PI Hopf Algebras and Irreducible Modules of Maximal Dimension*

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We prove that a Noetherian Hopf algebra of finite global dimension possesses further attractive homological properties, at least when it satisfies a polynomial identity. This applies in particular to quantized enveloping algebras and to quantized function algebras at a root of unity, as well as to classical enveloping algebras in positive characteristic. In all three cases we show that these algebras are Auslander-regular and Macaulay. We derive representation theoretic consequences concerning the coincidence of the non-Azumaya and singular loci for each of the above three classes of algebras.

INTRODUCTION

This paper has two major themes. The first, which occupies Sections 1 and 2, is a contribution to a strand of research in the homological algebra of noncommutative Noetherian rings which perhaps began with [5], and

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whose most notable achievement is [45]. The underlying idea of all this work is that a Noetherian ring all of whose irreducible modules exhibit similar homological behaviour is "smooth" in some sense. Here this point of view is tested for Noetherian Hopf algebras. Our starting point is the observation of Lorenz and Lorenz [34, Sect. 2.4] that the global dimension of any Hopf algebra is just the projective dimension of its trivial module. The main result we deduce along this line in Section 1 (Corollary 1.8) has as a special case:

THEOREM A. *Let H be a Hopf algebra of finite global dimension which is a finite module over its affine center. Then H is Auslander-regular and Macaulay, and is thus a finite direct sum of prime rings.*

The terminology used in Theorem A is recalled in (1.8). In Section 2 we apply the foregoing ideas to quantized enveloping algebras and to quantized function algebras. References for the definitions of these algebras are given in (2.1) and (2.4). We consider here both the case of a generic parameter q, yielding the algebras $U_q(\mathfrak{g})$ and $\mathcal{O}_q[G]$, and the case where q specializes to a primitive lth root of unity ϵ, giving algebras $U_\epsilon[\mathfrak{g}]$ and $\mathcal{O}_\epsilon[G]$. Here, $\mathfrak{g}$ is a finite dimensional complex semisimple Lie algebra, and G is the connected, simply connected, semisimple Lie group with Lie algebra $\mathfrak{g}$. As usual, we require l to be odd, and prime to 3 if $\mathfrak{g}$ involves a factor of type G_2. We show, in Propositions 2.2 and 2.7 and Theorems 2.3 and 2.8:

THEOREM B. (i) *The algebras $U_q(\mathfrak{g})$, $U_\epsilon(\mathfrak{g})$, $\mathcal{O}_q[G]$, and $\mathcal{O}_\epsilon[G]$ are Noetherian domains of finite global dimension.*

(ii) *The algebras $U_\epsilon(\mathfrak{g})$ and $\mathcal{O}_\epsilon[G]$ are Auslander-regular and Macaulay with Krull and global dimensions equal to* $\dim \mathfrak{g}$.

We also deduce a conclusion identical to that of Theorem B(ii) for the enveloping algebra $U(\mathfrak{k})$ of any finite dimensional Lie algebra $\mathfrak{k}$ over a field of positive characteristic (Corollary 1.10).

Our second major theme is representation theoretic. We are interested in identifying, for an algebra Λ which is a finite module over its affine center Z, the irreducible modules of maximum dimension. When the base field is algebraically closed, the contractions of the annihilators of these modules to Z constitute the Azumaya locus, $\mathcal{A}$ say, a Zariski dense subset of $\max Z$. In favourable circumstances, specifically *when Λ is Auslander-regular and Macaulay and when* $\operatorname{codim}(\max Z \setminus \mathcal{A}) \geq 2$, *the set $\mathcal{A}$ coincides with the open set of non-singular points of* $\max Z$. In a graded setting this is a result of Le Bruyn [30, Proposition 5]. For the convenience of the reader we provide a self-contained, "un-graded" proof in Section 3, the result in question being Theorem 3.8.

In Section 4 we show that the results of Sections 2 and 3 can be combined to yield representation theoretic information for each of the classes of algebras $U_\epsilon(\mathfrak{g})$, $\mathcal{O}_\epsilon[G]$, and $U(\mathfrak{k})$. Our results in this section can be summarised as follows, where $U_\epsilon(\mathfrak{g})$ and $\mathcal{O}_\epsilon[G]$ are as described above, with the base field k algebraically closed, equal to $\mathbb{C}$ in cases (i) and (ii), and of characteristic $p > 0$ in (iii).

THEOREM C. *The locus of non-Azumaya points of the center coincides with the singular locus of the center for each of the following algebras*:

(i) $U_\epsilon(\mathfrak{g})$;

(ii) $\mathcal{O}_\epsilon[G]$;

(iii) *The enveloping algebra* $U(\mathfrak{g})$ *of the Lie algebra* $\mathfrak{g}$ *of a connected, simply connected, semisimple algebraic group G over k, provided that*

(a) *p is a good prime for G* (*see* (4.6)), *and*

(b) *G has no component of type* A_r *with* $r \equiv -1 \pmod p$.

These results are obtained as Theorems 4.3, 4.5, and 4.10. The first verifies a conjecture of De Concini and Kac [10, Conjecture 5.2(c)], and the third partially answers a question of Premet [41, Sect. 4.4, Question 2].

1. HOPF ALGEBRAS AND HOMOLOGY

We begin by developing some homological information about modules over a Hopf algebra H, leading to sufficient conditions for H to be Auslander-regular and Macaulay. The main result of the section is that these conclusions hold when H is fully bounded Noetherian (FBN) with finite global dimension and all irreducible H-modules are finite dimensional over the base field. Note that the FBN and irreducible module conditions will hold in case H is a finite module over its center and the center is an affine algebra over the base field. (An analogous Auslander–Gorenstein result is obtained for the case that H has finite injective dimension rather than finite global dimension.) With the help of known results, this theorem immediately applies to the enveloping algebra of any finite dimensional Lie algebra in positive characteristic. The background needed to apply the theorem to quantized enveloping algebras and quantized function algebras at roots of unity is developed in the following section.

1.0. Throughout the section, H will denote a Hopf algebra over a base field k, and H-modules will be left modules unless otherwise specified. All tensor products will be over k. Recall the standard actions by which tensor products and Hom-groups of H-modules A and B become H-modules (e.g., [27, p. 58]),

$$h.(a \otimes b) = \sum_{(h)} h_{(1)}.a \otimes h_{(2)}.b, \qquad (h.f)(a) = \sum_{(h)} h_{(1)}.\big[f\big(S(h_{(2)}).a\big)\big]$$

for $h \in H$, $a \in A$, $b \in B$, $f \in \mathrm{Hom}_k(A, B)$. In case the H-action on B is trivial, the second formula simplifies. In particular,

$$\begin{aligned}(h.f)(a) &= \sum_{(h)} \epsilon(h_{(1)})\big[f\big(S(h_{(2)}).a\big)\big] \\ &= \sum_{(h)} f\big(S\big(\epsilon(h_{(1)})h_{(2)}\big).a\big) = f(S(h).a)\end{aligned}$$

for $h \in H$, $a \in A$, $f \in A^*$. It is a standard fact that for any (left) H-modules V and W, the natural map $\theta\colon V \otimes W^* \to \mathrm{Hom}_k(W, V)$, given by $\theta(v \otimes f)(w) = f(w)v$, is an H-module map (e.g., [27, Proposition III.5.2]).

LEMMA 1.1. *Let A, B, C be H-modules. Then the natural adjoint isomorphism ϕ: $\mathrm{Hom}_k(A, \mathrm{Hom}_k(B, C)) \to \mathrm{Hom}_k(A \otimes B, C)$ restricts to an isomorphism*

$$\mathrm{Hom}_H(A, \mathrm{Hom}_k(B, C)) \xrightarrow{\cong} \mathrm{Hom}_H(A \otimes B, C).$$

Proof. Recall that ϕ sends any $f \in \mathrm{Hom}_k(A, \mathrm{Hom}_k(B, C))$ to the map $\bar{f} = \phi(f) \in \mathrm{Hom}_k(A \otimes B, C)$ such that $\bar{f}(a \otimes b) = f(a)(b)$ for $a \in A$, $b \in B$. One checks that f is H-linear if and only if $\bar{f}$ is—see the proof of [40, Lemma 4]. ∎

1.2. If H has a bijective antipode, one can also prove Lemma 1.1 by showing that ϕ is H-linear and using the fact that $\mathrm{Hom}_H(X, Y)$ equals the subspace of H-invariants of $\mathrm{Hom}_k(X, Y)$ (cf. [13, 3.1]).

An immediate consequence of Lemma 1.1 is that $A \otimes B$ is a projective H-module for any H-modules A and B such that A is projective (just use that $\mathrm{Hom}_H(A, -)$ and $\mathrm{Hom}_k(B, -)$ are exact functors). Similarly, $\mathrm{Hom}_k(B, C)$ is an injective H-module for any H-modules B and C such that C is injective. Both of these facts are derived (in the presence of an additional but unused assumption) in [40, Lemma 5(b), (d)].

The referee has pointed out that the following proposition can also be obtained from Lemma 1.1 via the Grothendieck spectral sequence for composite functors.

PROPOSITION 1.3. $\mathrm{Ext}^i_H(W \otimes V, X) \cong \mathrm{Ext}^i_H(W, \mathrm{Hom}_k(V, X))$ *for all i and all H-modules V, W, X.*

Proof. Let $\cdots \to P_i \to \cdots \to P_1 \to P_0 \to W \to 0$ be an H-projective resolution for W. Then

$$\cdots \to P_i \otimes V \to \cdots \to P_1 \otimes V \to P_0 \otimes V \to W \otimes V \to 0$$

is an H-projective resolution for $W \otimes V$. From Lemma 1.1, we get a commutative diagram of complexes as follows:

$$\begin{array}{ccccc}
\mathrm{Hom}_H\big(P_0, \mathrm{Hom}_k(V, X)\big) & \to \cdots \to & \mathrm{Hom}_H(P_i, \mathrm{Hom}_k(V, X)) & \to & \cdots \\
\cong\downarrow & & \downarrow\cong & & \\
\mathrm{Hom}_H(P_0 \otimes V, X) & \to \cdots \to & \mathrm{Hom}_H(P_i \otimes V, X) & \to & \cdots
\end{array}$$

The homology of the top row is $\mathrm{Ext}_H^\bullet(W, \mathrm{Hom}_k(V, X))$, while the homology of the bottom is $\mathrm{Ext}_H^\bullet(W \otimes V, X)$. ∎

1.4. For an H-module V we denote the projective and injective dimensions of V by $\mathrm{p.dim}_H(V)$ and $\mathrm{i.dim}_H(V)$, and we write $\mathrm{gl.dim}(H)$ for the left global dimension of H. Note that it is a routine exercise to show that the left and right global dimensions of a Hopf algebra H are equal if its antipode S is bijective, and the same equality holds for any Noetherian ring [42, Corollary 9.23]. Part (c) of the next corollary is noted in [34, Sect. 2.4] and (under an extra hypothesis) in [40, Corollary 4]. Here ${}_Hk$ denotes k equipped with the trivial H-action. The referee has pointed out that in the case when H has a bijective antipode, part (c) can be derived from [7, Theorem X.6.2], which then also shows that the global dimension of H coincides with its Hochschild dimension.

COROLLARY. (a) $\mathrm{p.dim}_H(W \otimes V) \leq \mathrm{p.dim}_H(W)$ *for all H-modules V, W.*

(b) $\mathrm{i.dim}_H(\mathrm{Hom}_k(V, X)) \leq \mathrm{i.dim}_H(X)$ *for all H-modules V, X.*

(c) $\mathrm{l.gl.dim}(H) = \mathrm{p.dim}({}_Hk)$.

Proof. Parts (a) and (b) are immediate from the proposition. Taking $W = k$ in (a), we obtain $\mathrm{p.dim}_H(V) \leq \mathrm{p.dim}({}_Hk)$ for all H-modules V; part (c) follows. ∎

COROLLARY 1.5 [13, 3.1]. *If ${}_Hk$ is projective within the category of finite dimensional H-modules, then all finite dimensional H-modules are semisimple.*

Proof. For all finite dimensional H-modules V and X, we have $\mathrm{Ext}_H^1(k, \mathrm{Hom}_k(V, X)) = 0$ by the hypothesis on k. Therefore $\mathrm{Ext}_H^1(V, X) = 0$ by Proposition 1.3. ∎

While the menagerie of noncommutative Noetherian algebras of finite global dimension includes many exotic beasts, a persistent theme in work over the past fifteen years (e.g., [4, 45, 50]) has been that if homological behaviour is "homogeneous" across all the irreducible modules for the algebra then its structure is much more constrained, and indeed resembles a regular commutative ring in important respects. That this heuristic applies also to at least some Noetherian Hopf algebras is the theme of the rest of Section 1. We begin with the proof of homogeneity.

PROPOSITION 1.6. *Assume that* $\mathrm{p.dim}({}_H k) < \infty$. *If V is any nonzero finite dimensional H-module, then* $\mathrm{p.dim}_H(V) = \mathrm{p.dim}_H(k)$.

Proof. Let $n = \mathrm{p.dim}({}_H k)$, and suppose that $\mathrm{p.dim}_H(V) < n$. The k-dual V^* of V is an H-module as in (1.0), and Corollary 1.4(a) shows that $\mathrm{p.dim}_H(V \otimes V^*) < n$.

Fix a basis $\{v_i\}$ of V and the corresponding dual basis $\{w_i\}$ of V^*. Define the "coevaluation map" $\delta\colon k \to V \otimes V^*$ by $\delta(\alpha) = \sum_i \alpha v_i \otimes w_i$. By [27, Proposition III.5.3(b)], δ is an H-module homomorphism, and so we have an exact sequence

$$0 \to k \xrightarrow{\delta} V \otimes V^* \to B \to 0$$

of finite dimensional H-modules. Since $\mathrm{p.dim}_H(V \otimes V^*) < n = \mathrm{p.dim}({}_H k)$, standard properties of projective dimension (e.g., [26, Part III, Theorem 2(3)]) imply that $\mathrm{p.dim}_H(B) = n + 1$. However, this contradicts Corollary 1.4(c), and the proposition is proved. ∎

For an alternate proof in the case that H is left Noetherian, see (1.12) below.

1.7. The first consequence of Proposition 1.6 is well known. It follows, for example, from the fact that every finite dimensional Hopf algebra is a Frobenius algebra [28, Remark, p. 85; 37, Theorem 2.1.3].

COROLLARY. *A finite dimensional Hopf algebra of finite global dimension is semisimple.*

Proof. If H is such an algebra then $\mathrm{p.dim}_H(V) = \mathrm{p.dim}_H(H) = 0$ for all finite dimensional H-modules V. ∎

1.8. A Noetherian ring R of finite global dimension is *Auslander-regular* if, for every finitely generated (right or left) R-module M and every submodule N of $\mathrm{Ext}^j_R(M, R)$, one has $\mathrm{Ext}^i_R(N, R) = 0$ for all non-negative integers $i < j$. As in [45], a *Macaulay ring* R is one for which $j(M) + \mathrm{K.dim}(M) = \mathrm{K.dim}(R)$ for every nonzero finitely generated R-module M, where

$$j(M) = \min\{j \geq 0 \mid \operatorname{Ext}^j_R(M, R) \neq 0\},$$

and K.dim(M) denotes the Krull dimension of M.

COROLLARY. *Let H be an FBN Hopf algebra of finite global dimension n, all of whose irreducible modules are finite dimensional over k. Then H is an Auslander-regular, Macaulay ring of Krull dimension n. Furthermore, H is a semiprime ring, and if H is PI, then H is a finite direct sum of prime rings.*

Proof. Proposition 1.6 shows that, in the language of [45, Sect. 5], H is smooth. The homological conclusions thus follow from [50, Theorem 3.9]. The remaining conclusions follow from [45, Propositions 5.2(ii) and 5.4(i)]. ∎

1.9. It appears to be an open question whether some version of Corollary 1.8 holds for arbitrary Noetherian Hopf algebras of finite global dimension. For instance, are they Auslander-regular and semiprime? Are they finite direct sums of prime rings? The example of enveloping algebras $U(\mathfrak{g})$ in characteristic zero shows that one cannot expect Noetherian Hopf algebras of finite global dimension to be Macaulay in general. Instead, one could ask for the "CM" property, which is defined like the Macaulay property but with Krull dimension replaced by Gelfand−Kirillov dimension. However, although this takes care of enveloping algebras it cannot encompass the case of Noetherian group algebras, since for these the Gelfand−Kirillov dimension is infinite in general. Probably the correct approach is to think of the Macaulay condition as requiring that $\delta(M) = \text{gl.dim}(R) - j(M)$ has the properties of an exact partitive dimension function.

1.10. The enveloping algebra of a Lie algebra $\mathfrak{g}$ of finite dimension n has global dimension n by [7, Theorem XIII.8.2]. When the ground field k has positive characteristic, $U(\mathfrak{g})$ is a finite module over its affine center [22] and so is a PI ring with all irreducible modules finite dimensional over k. Thus the previous corollary specialises to yield the following result, which can also be obtained using filtered and graded techniques in the style of [32].

COROLLARY. *Let $\mathfrak{g}$ be a Lie algebra of finite dimension n over a field of positive characteristic. Then $U(\mathfrak{g})$ is Auslander-regular and Macaulay of Krull and global dimension n.* ∎

In the remainder of the section, we develop an analog of Corollary 1.8 for the case when H has finite injective dimension rather than finite global dimension, the aim being to replace the Auslander-regular conclusion with the Auslander-Gorenstein condition.

LEMMA 1.11. *Given $n \geq 0$, the following conditions are equivalent*:

(a) $\operatorname{Ext}_H^n(k, H) \neq 0$.

(b) $\operatorname{Ext}_H^n(V, H) \neq 0$ *for some finite dimensional H-module V*.

(c) $\operatorname{Ext}_H^n(V, H) \neq 0$ *for all nonzero finite dimensional H-modules V*.

Proof. Let V be a nonzero finite dimensional H-module. The natural map $H \otimes V^* \to \operatorname{Hom}_k(V, H)$ is a left H-module map, and it is bijective because V is finite dimensional. By the Fundamental Theorem of Hopf Modules, $H \otimes V^*$ is a free H-module of finite rank. (If $f_1, \ldots, f_r$ is a k-basis for V^*, then $1 \otimes f_1, \ldots, 1 \otimes f_r$ is an H-basis for $H \otimes V^*$.) Thus, $\operatorname{Hom}_k(V, H) \cong H^r$ for some positive integer r. By Proposition 1.3, $\operatorname{Ext}_H^n(V, H) \cong \operatorname{Ext}_H^n(k, H)^r$, and therefore $\operatorname{Ext}_H^n(V, H) \neq 0$ if and only if $\operatorname{Ext}_H^n(k, H) \neq 0$. ∎

1.12. Here is an alternate proof of Proposition 1.6 in case H is left Noetherian. Let $n = \text{p.dim}({}_H k) = \text{l.gl.dim}(H)$. Since H is left Noetherian, $\operatorname{Ext}_H^n(k, H) \neq 0$, and so $\operatorname{Ext}_H^n(V, H) \neq 0$ by Lemma 1.11. Therefore $\text{p.dim}_H(V) = n$. ∎

1.13. Question: What is the H-module structure of $\operatorname{Hom}_k(V, H)$ for arbitrary V? In view of the finite dimensional case (see the proof of Lemma 1.11), one might guess it to be a direct product of copies of H. However, that is not correct in general. For example, assume that H is left Noetherian with $\text{l.gl.dim}(H) = n < \infty$. Then $\operatorname{Ext}_H^n(k, H) \neq 0$. On the other hand, $\operatorname{Ext}_H^n(k, \operatorname{Hom}_k(H, H)) \cong \operatorname{Ext}_H^n(H, H) = 0$ by Proposition 1.3, and therefore ${}_H H$ cannot be isomorphic to a direct summand of ${}_H \operatorname{Hom}_k(H, H)$.

1.14. Recall that a Noetherian ring R of finite injective dimension n is *Artin–Schelter Gorenstein* if, for all irreducible (right or left) R modules M and all $i \geq 0$, the group $\operatorname{Ext}_R^i(M, R)$ is Artinian, and is 0 unless $i = n$. It is easy to prove that if H is a Hopf algebra with a bijective antipode then its left and right injective dimensions coincide. It follows from Lemma 1.11 and [3, Theorems A, D] that if H is an FBN Hopf algebra over an uncountable base field, or if H is a Noetherian PI Hopf algebra, one can determine $\text{i.dim}(H)$ by the vanishing of $\operatorname{Ext}_H^i(k, H)$.

THEOREM. *Let H be an FBN Hopf algebra, all of whose irreducible modules are finite dimensional over k. If H has finite injective dimension, then H is Artin–Schelter Gorenstein. Moreover, if H is PI with finite injective dimension, then H is an Auslander–Gorenstein, Macaulay ring of Krull dimension n.*

Proof. Let $n = \text{i.dim}({}_H H)$. By [21, Proposition 8], there exists an irreducible H-module V such that $\operatorname{Ext}_H^n(V, H) \neq 0$. Since all irreducible

H-modules are finite dimensional by hypothesis, Lemma 1.11 implies that $\operatorname{Ext}^n_H(X, H) \neq 0$ for all irreducible H-modules X. Hence, H is injectively smooth, and the results follow from [50, Theorem 2.7; 45, Theorem 3.10]. ∎

1.15. It is natural to expect that H is Auslander–Gorenstein in Theorem 1.14 without the additional PI hypothesis; but perhaps it is more important to mention two other themes suggested by this result. First, it seems plausible that *every* FBN Hopf algebra which is affine over its base field k satisfies a PI. (It is worth noting in this connection the recent result of Amitsur and Small [1] that every FBN affine k-algebra satisfies a PI when k is algebraically closed.) Second, it appears to be possible that, under the hypotheses of Theorem 1.14, H *always* has finite injective dimension. If true, this would be a notable generalisation of the dimension zero case mentioned in (1.7). We can even wonder whether this homological finiteness might be a property of *all* Noetherian Hopf algebras.

2. GLOBAL DIMENSION OF QUANTIZED ENVELOPING ALGEBRAS AND QUANTIZED FUNCTION ALGEBRAS

Our goal in this section is to apply Corollary 1.8 to quantized enveloping algebras $U_\epsilon(\mathfrak{g})$ and quantized function algebras $\mathcal{O}_\epsilon[G]$ when ϵ is a root of unity. (References for descriptions of these algebras are given in (2.1) and (2.4).) That these algebras are Noetherian domains, finite over affine central subalgebras, follows readily from known results; somewhat more effort is required to prove finiteness of their global dimensions.

2.1. Let $\mathfrak{g}$ be a finite dimensional semisimple complex Lie algebra, and let $U_q(\mathfrak{g})$ denote the corresponding quantized enveloping algebra over a field k of characteristic zero, where q denotes an indeterminate. For our purposes, it suffices that k contain $\mathbb{Q}(q)$ and appropriate roots of q. Here the role of $\mathfrak{g}$ is just to record the associated Cartan matrix $C = (a_{ij})_{n\times n}$, and $U_q(\mathfrak{g})$ is the k-algebra with generators $E_i, F_i, K_i^{\pm 1}$ $(1 \leq i \leq n)$ and relations as listed in, e.g., [9, Sect. 9.1A; 13, Sect. 9.1; 24, Sect. 5.1.1]. We shall also consider the algebra $U_\epsilon(\mathfrak{g})$ where ϵ is a primitive lth root of unity; we take l to be odd, and prime to 3 in case $\mathfrak{g}$ involves a factor of type G_2. This algebra is defined by constructing a $\mathbb{Q}[q, q^{-1}]$-form U_A of $U_q(\mathfrak{g})$ and then factoring U_A by the ideal generated by the lth cyclotomic polynomial $p_l(q) \in \mathbb{Q}[q]$; see, e.g., [10, Sect. 1.5; 13, Sects. 12.1, 19.1; 35, Sect. 8.1]. Thus the basic form of $U_\epsilon(\mathfrak{g})$ is an algebra over $\mathbb{Q}(\epsilon)$; the scalars can of course be extended to any field $k' \supseteq \mathbb{Q}(\epsilon)$, for instance by forming $U_A \otimes_{\mathbb{Q}[q, q^{-1}]} k'$ where q acts as ϵ on k'. Both $U_q(\mathfrak{g})$ and $U_\epsilon(\mathfrak{g})$ have Hopf algebra structures (op. cit.). Note that these are the *simply connected* forms of $U_q(\mathfrak{g})$ and $U_\epsilon(\mathfrak{g})$, inasmuch as the group $\langle K_1, \dots, K_n \rangle$ identifies with the lattice of weights P of $\mathfrak{g}$. Variants (for which some aspects of the theory are more complex) can be defined for any lattice lying between P and the root lattice Q.

PROPOSITION 2.2. *The algebras $U_q(\mathfrak{g})$ and $U_\epsilon(\mathfrak{g})$ are Auslander-regular Noetherian domains of finite global dimension.*

Proof. By [10, Proposition 1.7] or [13, Proposition 10.1 and Remark], there is a sequence of algebras $U_q(\mathfrak{g}), U^{(0)}, \dots, U^{(N)}$ each of which is the associated graded ring of the previous algebra with respect to a $\mathbb{Z}^+$-filtration, and such that the last algebra $U^{(N)}$ is an iterated skew polynomial ring starting from a Laurent polynomial ring over k. Reducing to the subalgebra U_A and specializing q to ϵ yields a corresponding sequence of algebras starting with $U_\epsilon(\mathfrak{g})$ [10, Proposition 1.7; 13, Sect. 20.2]. Now iterated skew polynomial rings of this form are Noetherian domains of finite global dimension (e.g., [36, Theorems 1.2.9 and 7.5.3]), and thus it follows from standard filtered/graded techniques [36, Proposition 1.6.6, Theorem 1.6.9, and Corollary 7.6.18] that $U_q(\mathfrak{g})$ and $U_\epsilon(\mathfrak{g})$ are Noetherian domains of finite global dimension. Now Laurent polynomial rings $k[z_1^{\pm 1}, \dots, z_t^{\pm 1}]$ are Auslander-regular (being commutative noetherian of finite global dimension), and Auslander-regularity is preserved in skew polynomial extensions [16, Theorem 4.2]. Hence, we conclude from a final filtered/graded lifting via [2, Theorem 3.9] that $U_q(\mathfrak{g})$ and $U_\epsilon(\mathfrak{g})$ are Auslander-regular. ∎

THEOREM 2.3. *The Hopf algebra $U_\epsilon(\mathfrak{g})$ is Macaulay with Krull and global dimensions equal to* $\dim \mathfrak{g}$.

Proof. It is known that $U_\epsilon(\mathfrak{g})$ is a finite module over an affine central subring Z_0 of Krull dimension $n = \dim \mathfrak{g}$ [13, Theorem 19.1; 9, Proposition 9.2.7 and following paragraph]. Hence, $U_\epsilon(\mathfrak{g})$ is a PI algebra of Krull dimension n, with all irreducible modules finite dimensional over k'. Since it is also Noetherian with finite global dimension by the previous proposition, Corollary 1.8 applies to give the stated conclusions. ∎

2.4. We now turn to the quantized algebras of functions $\mathcal{O}_q[G]$ and $\mathcal{O}_\epsilon[G]$. Here G is a label referring to the connected, simply connected, semisimple Lie group with Lie algebra $\mathfrak{g}$. As in [24, Sect. 9.1.1], we take $\mathcal{O}_q[G]$ to be the sub-Hopf-algebra of the Hopf dual $U_q(\mathfrak{g})^\circ$ spanned by the coordinate functions of the standard highest weight modules $V(\mu)$. The Hopf algebra $\mathcal{O}_\epsilon[G]$ is obtained from an analogous sub-Hopf-algebra $\mathcal{O}_A$ of the Hopf dual of the $\mathbb{Q}[q, q^{-1}]$-form U_A of $U_q(\mathfrak{g})$ on specializing modulo the ideal generated by $p_l(q)$ [12, Sects. 4.1 and 6.1; 35, Sects. 7.1 and 8.17]. In particular, $\mathcal{O}_\epsilon[G]$ embeds in $U_\epsilon(\mathfrak{g})^\circ$ by [12, Lemma 6.1]. We shall also need the form of $\mathcal{O}_\epsilon[G]$ defined over an (algebraically closed) field $k' \supseteq \mathbb{Q}(\epsilon)$ by extending scalars. Alternately, as in [14, Sects. 1.4 and 1.6] one can define a $k'[q, q^{-1}]$-form of $\mathcal{O}_A$ and factor modulo the ideal generated by $q - \epsilon$ to obtain $\mathcal{O}_\epsilon[G]$.

2.5. Write $\breve{U}_q(\mathbf{b}^{\pm})$ for the simply connected versions of the Borel subalgebras of $U_q(\mathfrak{g})$. That is, for example,

$$\breve{U}_q(\mathbf{b}^+) = U_q(\mathbf{n})^+ \left[t_\alpha \mid \alpha \in P(\pi)\right],$$

in the notation of [24], where $P(\pi)$ denotes the lattice of weights given by the Cartan data defining $\mathfrak{g}$ and $U_q(\mathbf{n}^+) = k\langle E_\alpha \mid \alpha \in \pi \rangle$ is the "uppertriangular" part of $U_q(\mathfrak{g})$. For details, see [24, Sect. 3.2.10]. The algebras $U_q(\mathbf{b}^{\pm})$ are defined in the same way, but using the *root* lattice $Q(\pi)$ of $\mathfrak{g}$.

The embeddings of $U_q(\mathbf{b}^{\pm})$ in $U_q(\mathfrak{g})$ yield Hopf algebra homomorphisms of their duals. Let Ψ^+ and Ψ^- denote the restrictions of these homomorphisms to $\mathcal{O}_q[G]$, and let $\mathcal{O}_q[B^+]$ and $\mathcal{O}_q[B^-]$ denote the images of Ψ^+ and Ψ^- in the Hopf duals of $U_q(\mathbf{b}^+)$ and $U_q(\mathbf{b}^-)$. By [24, Corollary 9.2.12] there are isomorphisms of Hopf algebras

$$\mathcal{O}_q[B^-] \cong \breve{U}_q(\mathbf{b}^-) \qquad \text{and} \qquad \mathcal{O}_q[B^+] \cong \breve{U}_q(\mathbf{b}^+).$$

Moreover, composing the comultiplication Δ on $\mathcal{O}_q[G]$ with $\Psi^- \otimes \Psi^+$ and finally with the above isomorphisms gives an algebra (but not Hopf algebra) homomorphism

$$\Psi\colon \mathcal{O}_q[G] \to \breve{U}_q(\mathbf{b}^-) \otimes \breve{U}_q(\mathbf{b}^+)$$

which is *injective* by [24, Lemma 9.2.13].

let $\mathcal{C}$ denote the multiplicative set generated by

$$\left\{c^{\omega_i}_{\omega_i\omega_i} \,\middle|\, 1 \le i \le l\right\} \subseteq \mathcal{O}_q[G]$$

in the notation of [24, Sect. 9.1.1] (where these elements are assumed to be normalised in the manner discussed in [24, Sect. 9.1.10]). By [24, Lemma 9.1.10(i)], $\mathcal{C}$ is an Ore set in $\mathcal{O}_q[G]$. Write t_i for $t_{\omega_i} \in U_q(\mathfrak{g})$, $i = 1, \dots, l$. Then $\Psi(c^{\omega_i}_{\omega_i\omega_i}) = t_i^{-1} \otimes t_i$ for $i = 1, \dots, l$. Since these are invertible elements of the image, Ψ extends to a homomorphism (which we also denote by Ψ) from $\mathcal{O}_q[G]\mathcal{C}^{-1}$ into $\breve{U}_q(\mathbf{b}^-) \otimes \breve{U}_q(\mathbf{b}^+)$. Let A be the subalgebra of $\breve{U}_q(\mathbf{b}^-) \otimes \breve{U}_q(\mathbf{b}^+)$ generated by the antidiagonal copy of the torus, $\langle t_i^{-1} \otimes t_i \,|\, 1 \le i \le l\rangle$ together with $\{e_i \otimes 1,\ 1 \otimes f_i \,|\, 1 \le i \le l\}$. Thus A is the skew group ring of a free abelian group of rank l over the coefficient ring $U_q(\mathbf{n}^-) \otimes U_q(\mathbf{n}^+)$. By [24, Proposition 9.2.14],

$$\Psi\left(\mathcal{O}_q[G]\mathcal{C}^{-1}\right) = A. \tag{1}$$

2.6. The set-up outlined in (2.5) can be obtained also at the level of forms over $\mathbb{Q}[q, q^{-1}]$. In this setting the analogue of (1) is [12, Lemma 4.3 and Theorem 4.6]. Then, as noted in [12, Remark after Proposition 6.4], one can factor by the appropriate cyclotomic polynomial to get

$$\Psi\big(\mathcal{O}_\epsilon[G]\mathcal{C}^{-1}\big) = A_\epsilon,$$

where A_ϵ is the corresponding skew group ring over $U_\epsilon(\mathbf{n}^-) \otimes U_\epsilon(\mathbf{n}^+)$. (In this case one can even take $\mathcal{C}$ in the center of $\mathcal{O}_\epsilon[G]$, by replacing the generators of $\mathcal{C}$ by their lth powers.)

PROPOSITION 2.7. *The algebras $\mathcal{O}_q[G]$ and $\mathcal{O}_\epsilon[G]$ are Noetherian domains of finite global dimension.*

Proof. The Noetherian and domain conclusions are given by [24, Lemma 9.1.9(i) and Proposition 9.2.2; 12, Theorems 7.2 and 7.4]. The algebras A and A_ϵ of (2.5) and (2.6) have finite global dimension for reasons similar to those utilized in Proposition 2.2, as follows. Restricting the filtrations mentioned to $U_q(\mathbf{n}^\pm)$ and $U_\epsilon(\mathbf{n}^\pm)$, we find that each of the latter algebras begins a finite sequence in which each algebra is the associated graded algebra of the previous one with respect to a $\mathbb{Z}^+$-filtration, and in which the final algebra is an iterated skew polynomial ring starting from a Laurent polynomial ring over a field. We immediately obtain similar sequences for the algebras $U_q(\mathbf{n}^-) \otimes U_q(\mathbf{n}^+)$ and $U_\epsilon(\mathbf{n}^-) \otimes U_\epsilon(\mathbf{n}^+)$, and it follows as in Proposition 2.2 that these algebras have finite global dimension. Since A and A_ϵ are skew group rings of free abelian groups over these coefficient rings, i.e., iterated skew-Laurent extensions, the claim follows [36, Theorem 7.5.3].

In particular, thanks to the isomorphisms Ψ of (2.5) and (2.6), $\mathcal{O}_q[G]\mathcal{C}^{-1}$ and $\mathcal{O}_\epsilon[G]\mathcal{C}^{-1}$ have finite global dimension. Write M for the trivial module over either $\mathcal{O}_q[G]$ or $\mathcal{O}_\epsilon[G]$. Now the counit γ applied to any element $c^\lambda_{\lambda,\lambda}$ of $\mathcal{C}$ yields (in both cases) 1, as is clear from the definitions. Thus $\mathcal{O}_q[G]\mathcal{C}^{-1} \otimes_{\mathcal{O}_q[G]} M = M$, and similarly for the case where q is specialized to ϵ. We thus have a finite projective resolution of M as $\mathcal{O}_q[G]\mathcal{C}^{-1}$- or as $\mathcal{O}_\epsilon[G]\mathcal{C}^{-1}$-module. However, these two localizations are flat as $\mathcal{O}_q[G]$- and as $\mathcal{O}_\epsilon[G]$-modules, respectively, so we have found a finite flat resolution of M in each case. Since $\mathcal{O}_q[G]$ and $\mathcal{O}_\epsilon[G]$ are Noetherian, it follows that M has finite projective dimension in each case (e.g., [36, Sect. 7.1.5]). The finiteness of the global dimension follows from Corollary 1.4(c). ∎

2.8. Note that $\mathcal{O}_\epsilon[G]$ is a projective module of finite rank over a copy of $\mathcal{O}[G]$ in its center [12, Theorem 7.2 and Proposition 6.4], so that

$$\text{K.dim}(\mathcal{O}_\epsilon[G]) = \text{GK.dim}(\mathcal{O}_\epsilon[G]) = \text{K.dim}(\mathcal{O}[G]) = \dim(G).$$

It follows also that $\mathcal{O}_\epsilon[G]$ is PI with all irreducible modules finite dimensional over k'. Hence, Corollary 1.8 when combined with Proposition 2.7 yields the

THEOREM. *The Hopf algebra $\mathcal{O}_\epsilon[G]$ is Auslander-regular and Macaulay of global, Gelfand–Kirillov, and Krull dimensions equal to the dimension of G.* ∎

3. THE AZUMAYA LOCUS

Suppose that Λ is a prime Noetherian ring finitely generated as a module over its center Z, which is in turn an affine algebra over an algebraically closed base field k. Problem: Which maximal ideals m of Z arise (as Z-annihilators) from irreducible Λ-modules of maximal k-dimension? As we shall note, this is equivalent to an alternate problem: For which $m \in \max Z$ is Λ_m an Azumaya algebra over Z_m? The latter formulation is more convenient to work with, since it does not depend on the base field, nor does it require the center to be affine. It turns out that provided Λ is Auslander-regular and Macaulay, and provided the non-Azumaya locus in $\max Z$ has codimension at least 2, this locus coincides with the locus of singular points in $\max Z$. This result was proved for graded algebras using sheaf-theoretic methods by Le Bruyn [30, Proposition 5]. For the reader's convenience, we state and prove the result (Theorem 3.8) in a purely ring-theoretic setting and shorn of the "graded" hypothesis.

To begin, we assemble some standard facts which show that the two problems mentioned above are in fact equivalent. Recall that a ring R with center C is called an *Azumaya algebra over C* provided R is projective as a module over $R \otimes_C R^{\mathrm{op}}$ (cf. [43, Theorem 5.3.24] for several equivalent conditions). In particular, R is Azumaya over C if and only if R is a finitely generated projective C-module and the natural map $R \otimes_C R^{\mathrm{op}} \to \mathrm{End}_C(R)$ is an isomorphism.

PROPOSITION 3.1. *Let Λ be a prime Noetherian ring which is module-finite over its center Z, and assume that Z is an affine algebra over an algebraically closed field k.*

(a) *The maximum k-dimension of irreducible Λ-modules equals the PI-degree of Λ.*

(b) *Let S be an irreducible Λ-module, $P = \mathrm{ann}_\Lambda(S)$, and $m = P \cap Z$. Then $\dim_k(S) = \text{PI-deg}(\Lambda)$ if and only if Λ_m is Azumaya over Z_m.*

Proof. Note that if P is any primitive ideal of Λ, then Λ/P is a finite dimensional simple algebra over the field Z/m, where $m = P \cap Z$. However, $Z/m = k$ by our hypotheses, and so $\Lambda/P \cong M_n(k)$ where n is the dimension of the unique irreducible (Λ/P)-module.

(a) Let d be the maximum dimension of the irreducible Λ-modules. Since Λ is module-finite over Z, which is commutative affine and therefore Jacobson, Λ is Jacobson (e.g., [36, Theorem 9.4.21]). Thus, by primeness, $J(\Lambda) = 0$. For any primitive ideal P of Λ, we have $\Lambda/P \cong M_n(k)$ for some $n \leq d$ by the remarks above, whence PI-deg$(\Lambda/P) \leq d$. Therefore PI-deg$(\Lambda) \leq d$. On the other hand, Λ has at least one irreducible module S of dimension d, and $\Lambda/\mathrm{ann}_\Lambda(S) \cong M_d(k)$. Thus PI-deg$(\Lambda) \geq$ PI-deg$(\Lambda/\mathrm{ann}_\Lambda(S)) = d$ (e.g., [36, Lemma 13.7.2]).

(b) Set $d =$ PI-deg$(\Lambda) =$ PI-deg(Λ_m).

If Λ_m is Azumaya over Z_m, then by [36, Proposition 13.7.11 and Theorem 13.7.14], $\Lambda_m/m\Lambda_m$ is simple with PI-degree d. However, $\Lambda_m/m\Lambda_m \cong \Lambda/m\Lambda$, and so $\Lambda/m\Lambda \cong M_d(k)$. Thus S, which is an irreducible $(\Lambda/m\Lambda)$-module, must have dimension d.

Conversely, if $\dim_k(S) = d$, then $\Lambda/P \cong M_d(k)$ and PI-deg$(\Lambda/P) = d$. Hence, P is a "regular" prime in the sense of PI-theory [36, 13.7.3], and 36, Proposition 13.7.5] implies that all primes of Λ_m are regular. Therefore by the Artin–Procesi Theorem [36, Theorem 13.7.14], Λ_m is Azumaya over its center. However, since Λ is prime, the center of Λ_m is just Z_m. ∎

3.2. Let Λ be a prime Noetherian ring which is module-finite over its center Z. Recall from [15, Theorem 1] that Z is necessarily noetherian. Define

$$\mathcal{A}_\Lambda := \{m \in \max Z \mid \Lambda_m \text{ is Azumaya over } Z_m\}$$

$$\mathcal{S}_\Lambda := \{m \in \max Z \mid Z_m \text{ is not regular}\}.$$

The set $\mathcal{A}_\Lambda$ is called the *Azumaya locus* of Λ, and $\mathcal{S}_\Lambda$ the *singular locus* of Λ (or of Z).

LEMMA 3.3. *Let Λ be a prime Noetherian ring, module-finite over its center Z, and suppose that* gl.dim $\Lambda < \infty$. *Then* $\mathcal{A}_\Lambda \subseteq \max Z \setminus \mathcal{S}_\Lambda$.

Proof. Let $m \in \max Z$ with Λ_m Azumaya over Z_m. Thus $m\Lambda_m$ is the (unique) maximal ideal of Λ_m [36, Proposition 13.7.9], and Λ_m is a finite free Z_m-module. A finite projective Λ_m-resolution of $\Lambda_m/m\Lambda_m$ thus affords a finite free Z_m-resolution of $\Lambda_m/m\Lambda_m$. Since Z_m/mZ_m is a direct summand of $\Lambda_m/m\Lambda_m$, it follows that Z_m/mZ_m has finite projective dimension and Z_m has finite global dimension. Thus $m \notin \mathcal{S}_\Lambda$. ∎

EXAMPLES 3.4. (i) Let k a field of characteristic $p > 0$, and $\Lambda = k[x][y; x(d/dx)]$. Then $Z = Z(\Lambda) = k[x^p, z]$ where $z = \prod_{i=0}^{p-1}(y - i)$. In this case, Z is regular, so $\mathcal{S}_\Lambda = \emptyset$. However, it is clear that $\mathcal{A}_\Lambda \neq \max Z$, since, e.g., $\langle x^p, z\rangle \notin \mathcal{A}_\Lambda$. Note that here Λ is Auslander-regular of dimension 2. In fact, it can be shown that $\mathcal{A}_\Lambda = \{m \in \max Z \mid x^p \notin m\}$, so that $\operatorname{codim}(\max Z \setminus \mathcal{A}_\Lambda) = 1$. Another way of saying this is that Z contains a height 1 prime $\langle x^p\rangle$ such that $\Lambda_{\langle x^p\rangle}$ is not Azumaya over $Z_{\langle x^p\rangle}$.

(ii) Let $\Lambda = U(\mathfrak{sl}_2(k))$ where k is a field of characteristic 2. If e, f, h is the standard basis for $\mathfrak{sl}_2(k)$, then $Z = Z(\Lambda) = k[e^2, f^2, h]$ and Λ is a free Z-module of rank 4. It follows that irreducible Λ-modules have dimension at most 2; in fact, the 2-dimensional irreducible Λ-modules are exactly those which are not annihilated by h. It follows that $\mathcal{A}_\Lambda = \{m \in \max Z \mid h \notin m\}$, so that as in the previous example, $\operatorname{codim}(\max Z \setminus \mathcal{A}_\Lambda) = 1$. Also as before, $\mathcal{A}_\Lambda \neq \max Z \setminus \mathcal{S}_\Lambda$, since Z is regular and so $\mathcal{S}_\Lambda = \emptyset$.

Examples 3.4 show that the reverse inclusion to that in Lemma 3.3 is false, even for Auslander-regular domains of dimension 2. We aim to show that the condition needed to ensure equality in Lemma 3.3 in the presence of Auslander-regularity is precisely that $\max Z \setminus \mathcal{A}_\Lambda$ should be smaller than codimension 1.

3.5. Let Λ be a prime Noetherian ring, module-finite over its center Z. It will be convenient to say that Λ is *height* 1 *Azumaya* if Λ_p is Azumaya over Z_p for all height 1 primes p of Z. In the language of [29, 31], Λ is *reflexive Azumaya* if it is height 1 Azumaya and Z is integrally closed. The relation of the height 1 Azumaya condition to the full Azumaya property is given in the following lemma, which is mentioned in [29, p. 251; 31, p. 66]. Since no proof is given in either reference, we provide one for the reader's convenience.

LEMMA 3.6. *Let Λ be a prime Noetherian ring, finitely generated and projective over its center Z. If Λ is height* 1 *Azumaya over Z, then it is Azumaya over Z.*

Proof. It suffices to show that Λ_m is Azumaya over Z_m for all $m \in \max Z$; hence, there is no loss of generality in assuming that Z is local. Now Λ is free over Z, say of rank r. Then $\Lambda \otimes_Z \Lambda^{\mathrm{op}}$ and $E := \operatorname{End}_Z(\Lambda)$ are both free over Z of the same rank, namely r^2.

Let $f\colon \Lambda \otimes_Z \Lambda^{\mathrm{op}} \to E$ be the natural map. For any height 1 prime p of Z, we have a commutative diagram as follows:

$$\begin{array}{ccc} \Lambda \otimes_Z \Lambda^{\mathrm{op}} & \xrightarrow{\ \ f\ \ } & E \\ {\scriptstyle \lambda_p}\downarrow & & \downarrow \\ \Lambda_p \otimes_{Z_p} \Lambda_p^{\mathrm{op}} & \xrightarrow{\ \ f_p\ \ } & \operatorname{End}_{Z_p}(\Lambda_p) \end{array}$$

Here λ_p is injective because $\Lambda \otimes_Z \Lambda^{\mathrm{op}}$ is torsionfree over Z, and f_p is an isomorphism by hypothesis. Consequently, f is injective. Now im f is free over Z of rank r^2, and so there is a Z-monomorphism $g\colon E \to E$ with $\operatorname{im} g = \operatorname{im} f$. For any height 1 prime p of Z, we have $\operatorname{im} g_p = \operatorname{im} f_p = E_p$; thus g_p is an isomorphism, and so det g is invertible in Z_p. Hence, det g is not contained in any height 1 prime. It follows that det g, and thus g, is invertible, that is, $\operatorname{im} f = \operatorname{im} g = E$. Therefore f is an isomorphism and Λ is Azumaya. ∎

PROPOSITION 3.7. *Let Λ be an Auslander–Gorenstein, Macaulay, prime Noetherian ring, module-finite over its center Z. If Λ is height* 1 *Azumaya over Z, then* $\mathcal{A}_\Lambda \supseteq \max Z \setminus \mathcal{S}_\Lambda$.

Proof. It follows from the Auslander–Gorenstein and Macaulay hypotheses that Λ is injectively homogeneous in the sense of [4]—see [45, Remark 3.11] or [52, Lemma 3.1]. Let $m \in \max Z$ with Z_m of finite global dimension. By [4, Theorem 3.4], Λ is a Cohen–Macaulay Z-module, and so Λ_m is a Cohen–Macaulay Z_m-module. However, a Cohen–Macaulay module over a regular local ring is free (e.g., by the Auslander–Buchsbaum depth theorem [6, Theorem 1.3.3]). Thus, by Lemma 3.6, Λ_m is Azumaya over Z_m, as required. ∎

Lemma 3.3 and Proposition 3.7 combine to give

THEOREM 3.8. *Let Λ be a prime Noetherian ring, module-finite over its center Z. If Λ is Auslander-regular and Macaulay, and if Λ_p is Azumaya over Z_p for all height* 1 *primes p of Z (that is,* $\max Z \setminus \mathcal{A}_\Lambda$ *has codimension at least* 2 *in* $\max Z$*), then* $\mathcal{A}_\Lambda = \max Z \setminus \mathcal{S}_\Lambda$. ∎

We remark on the side that the height 1 Azumaya hypothesis in Theorem 3.8 can be finessed in the presence of sufficient symmetry:

COROLLARY 3.9. *Let Λ be an Auslander-regular, Macaulay, prime Noetherian ring, module-finite over its center Z. If no height* 1 *prime of Z has a finite orbit under the restriction of* Aut Λ *to Z, then* $\mathcal{A}_\Lambda = \max Z \setminus \mathcal{S}_\Lambda$.

Proof. In view of [36, Proposition 13.7.4], there is a nonzero element $c \in Z$ such that all primes of $\Lambda[c^{-1}]$ are regular. By the Artin–Procesi Theorem, $\Lambda[c^{-1}]$ is Azumaya, and so Λ_p is Azumaya for all primes p of Z not containing c. Hence, there are at most finitely many height 1 primes p of Z for which Λ_p is not Azumaya. Since the set of such p is invariant under $(\operatorname{Aut} \Lambda)|_Z$, it must be empty by virtue of our hypotheses. Therefore Λ_p is Azumaya for all height 1 primes p of Z, and Theorem 3.8 applies. ∎

4. APPLICATIONS TO REPRESENTATION THEORY

4.1. Our aim in this section is to apply Theorem 3.8 to quantized enveloping algebras and to quantized function algebras at roots of unity, and to the enveloping algebras of semisimple Lie algebras in positive characteristic. In each of these three cases all but one of the hypotheses of the theorem have been shown to hold in earlier sections; the exception, which we need to consider here, is the codimension of the non-Azumaya locus. That is, we have to show that each algebra in these classes is Azumaya in codimension 1. We work over $\mathbb{C}$ in the first two cases.

4.2. We first consider the case of the quantized enveloping algebras $U_\epsilon = U_\epsilon(\mathfrak{g})$ over $\mathbb{C}$. We begin by listing, using [13] as basic reference, the properties of U_ϵ which we need. First, U_ϵ is a finite module over its affine center Z_ϵ, and is thus a Noetherian PI algebra whose PI degree is l^N (where N is the number of positive roots of $\mathfrak{g}$) [13, Theorems 19.1(ii) and 20.1(5)(6)]. The noncommutativity of $U_q(\mathfrak{g})$ induces a natural structure of Poisson algebra on Z_ϵ [13, Sect. 11.7]. There is a Poisson subalgebra Z_0 of Z_ϵ with the following properties: (I) Z_ϵ is a free Z_0-module of rank l^n (where $n = \operatorname{rank} \mathfrak{g}$), and U_ϵ is a free Z_0-module of rank l^{2N+n} [13, Proposition 20.2(ii) and Theorem 19.1(ii)]. (II) Z_0 is a Hopf subalgebra of U_ϵ and is thus the coordinate ring of a Poisson group H [13, Theorem 19.1(iii)]. Here, H is the dual of G in the algebraic Manin triple $(G \times G, H, G)$ of [13, Sect. 11.4, Example (2)]. Briefly,

$$H = \left\{(t^{-1}x^-, tx^+) \mid t \in T,\ x^\pm \in N^\pm\right\}$$

is the kernel of the homomorphism from $B^- \times B^+$ to T induced by projection of each factor to T, followed by multiplication. (Here T is a maximal torus of G and $B^\pm = N^\pm T$ are Borel subgroups.)

The Poisson structure on Z_0 induces a decomposition of $H = \operatorname{Max} Z_0$ as a disjoint union of symplectic leaves, as explained, for example, in [13, Sect. 11]. We view U_ϵ as a sheaf of finite dimensional algebras over Z_0: Thus, for $m \in \max Z_0$, the corresponding algebra is U_ϵ/mU_ϵ. Observe that the dimension over $\mathbb{C}$ of these algebras is constant, equal to l^{2N+n}, by (I) above. By [13, Corollary 11.8] if m_1 and m_2 belong to the same symplectic leaf of H, the algebras U_ϵ/m_1U_ϵ and U_ϵ/m_2U_ϵ are isomorphic. The symplectic leaves of H have been analysed in [13, Sect. 16]: There is an (infinite dimensional) group $\tilde{G}$ of analytic automorphisms of H whose orbits coincide with the leaves [13, Proposition 16.1]. Crucially, the action of $\tilde{G}$ on H can be analysed by virtue of the following remarkable result. Let $G^0 = N^-TN^+$ denote the big cell in G, and define

$$\sigma : H \to G^0 : (t^{-1}x^-, tx^+) \mapsto (x^-)^{-1}t^2x^+,$$

an unramified cover of G^0 of degree 2^n [13, Proposition 14.1]. De Concini, Kac, and Procesi show [11, Sects. 5.4, 5.5] that the vector fields on G^0 defined by the Chevalley generators of $\mathfrak{g}$ correspond, when lifted via σ to H, to derivations whose exponentials generate the group $\tilde{G}$. (This explains the terminology "quantum coadjoint action" for the action of $\tilde{G}$ on H.) From this it follows [13, Theorem 16.2] that each of the sets $\sigma^{-1}(\mathcal{C} \cap G^0)$, where $\mathcal{C}$ is a conjugacy class of a non-central element in G, is a $\tilde{G}$-orbit in H.

Denote by H_{reg} the union of the symplectic leaves of H of maximal dimension, and, correspondingly, let G_{reg} denote the set of regular elements of G, that is, the union of the conjugacy classes of maximal dimension (namely $2N$). By [13, Theorem 16.3(a)],

$$H_{\text{reg}} = \sigma^{-1}(G_0 \cap G_{\text{reg}}). \tag{2}$$

Let χ: max $Z_\epsilon \to H$ be the morphism induced by the inclusion of Z_0 in Z_ϵ, and recall from (3.2) that we write $\mathcal{A}_{U_\epsilon}$ for the (open) subset

$$\left\{m \in \max Z_\epsilon \,\middle|\, U_{\epsilon_m} \text{ is Azumaya over } Z_{\epsilon_m}\right\}$$

of Max Z_ϵ. In view of Proposition 3.1(b), [13, Theorem 24.1] shows that

$$\mathcal{A}_{U_\epsilon} \supseteq \chi^{-1}(H_{\text{reg}}). \tag{3}$$

Combining (2) and (3) and taking complements, we obtain

$$H \setminus \mathcal{A}_{U_\epsilon} \subseteq (\sigma \circ \chi)^{-1}\big(G_0 \setminus (G_0 \cap G_{\text{reg}})\big). \tag{4}$$

Now by [46, Theorem 1.3] the closed set $G \setminus G_{\text{reg}}$ has codimension 3 in G. Since σ and χ are finite morphisms it follows from (4) that

$$\operatorname{codim}_H(H \setminus \mathcal{A}_{U_\epsilon}) \geq 3. \tag{5}$$

As explained in (4.1), the inequality (5) is (more than) enough to permit us to apply Theorem 3.8, since the other hypotheses of that result have been confirmed to hold for $U_\epsilon(\mathfrak{g})$ in Proposition 2.2 and Theorem 2.3. The result we deduce proves [10, Conjecture 5.2(c)]. Further, since we are dealing with the $\mathbb{C}-$ form of $U_\epsilon(\mathfrak{g})$, the singular locus of Z_ϵ coincides with the set of maximal ideals over which irreducible $U_\epsilon(\mathfrak{g})$-modules have non-maximal dimension (cf. Proposition 3.1).

THEOREM 4.3. *Let $U_\epsilon(\mathfrak{g})$ be as stated in* (2.1), *with base field $k = \mathbb{C}$. Then the singular locus of its center Z_ϵ is equal to the non-Azumaya locus.* ∎

4.4. We next turn to the quantized function algebra $\mathcal{O}_\epsilon[G]$ over $\mathbb{C}$. As usual we assume that l is odd and that it is prime to 3 if there are G_2-components in $\mathfrak{g}$. Moreover, in order to apply [14] below, we must also assume that l is coprime to each a_j, where $\sum_{i=1}^n a_i \alpha_i$ is the expression for the highest root of $\mathfrak{g}$ as a linear combination of simple roots. There is a central sub-Hopf-algebra F_0 of $\mathcal{O}_\epsilon[G]$ which is isomorphic (as a Hopf algebra) to $\mathcal{O}[G]$ [12, Proposition 6.4], such that $\mathcal{O}_\epsilon[G]$ is a finitely generated projective F_0-module of rank $l^{\dim G}$ [12, Theorem 7.2]. Since the $\mathbb{Q}(\epsilon)$-form of $\mathcal{O}_\epsilon[G]$ is a factor of an integral form of $\mathcal{O}_q[G]$ by a principal ideal, the multiplication in this integral form induces a Poisson structure in the $\mathbb{Q}(\epsilon)$-form of F_0 [12, Theorem 8.5]; extending scalars, we obtain a Poisson structure in the $\mathbb{C}$-form of F_0. (This Poisson structure is the same as that discussed in [20, Appendix A], as noted in [12, Introduction].) The maximal torus T of G acts on $\mathcal{O}[G]$ by right and left translations, and these actions extend to actions of T by winding automorphisms on $\mathcal{O}_\epsilon[G]$ [12, Proposition 9.3].

The symplectic leaves of the Poisson Lie group G have been analysed by Hodges and Levasseur [20, Appendix A]. The leaves are permuted by an action of the torus T, and the T-orbits (which coincide, in fact, with those given by the winding automorphisms) are indexed by $W \times W$ where W is the Weyl group of G. Let (w_1, w_2) be an element of $W \times W$, and let $X_{(w_1, w_2)}$ denote the union of the corresponding orbit (cf. [12, Sect. 9.3]). The symplectic leaves contained in $X_{(w_1, w_2)}$ are all isomorphic, and have dimension

$$m(w_1, w_2) = l(w_1) + l(w_2) + s\left(w_1 w_2^{-1}\right); \tag{6}$$

here $l(w)$ denotes the length of $w \in W$, that is, the minimum length of an expression for w as a product of simple reflections, and $s(w)$ is the rank of the linear map $w - I$ on the Cartan subalgebra of $\mathfrak{g}$. By [24, Lemma A.1.18], $s(w)$ equals the minimum length of an expression for w as a product of arbitrary reflections in W. Hence, $m(w_1, w_2)$ is even for all $(w_1, w_2) \in W \times W$ (cf. [23, 8.11, Remark]), and (as follows from (8) below and [12, Corollary 7.3])

$$m(w_1, w_2) \leq 2N. \tag{7}$$

This upper bound is attained, for example (but not only), when $w_1 = w_2 = w_0$, the longest element of W.

Let $g \in G$ with $g \in X_{(w_1, w_2)}$. Let m_g be the corresponding maximal ideal of $\mathcal{O}[G] = F_0$, and let V be an irreducible $\mathcal{O}_\epsilon[G]$-module with $m_g V = 0$. By [14, Proposition 4.10] (cf. [14, Introduction; 12, Corollary 10.5 and Proposition 10.6]),

$$\dim_k V = l^{(1/2) m(w_1, w_2)}. \tag{8}$$

Recall that by [12, Corollary 7.3], $\mathcal{O}_\epsilon[G]$ has PI-degree l^N. Therefore it follows from (7), (8), and Proposition 3.1 that an element $g \in G$ belongs to a symplectic leaf of maximal dimension $2N$ if and only if $\mathcal{O}_\epsilon[G]$ is Azumaya over its center Z_ϵ at every maximal ideal $\hat{m}_g$ of Z_ϵ containing m_g. That is, the non-Azumaya locus $\mathcal{A}'$ of $\mathcal{O}_\epsilon[G]$ is given by

$$\mathcal{A}' = (\sigma\tau)^{-1}\left(\bigcup_{\substack{(w_1,w_2)\in W\times W\\ m(w_1,w_2)\leq 2N-2}} X_{(w_1,w_2)}\right), \tag{9}$$

where σ: spec $F_0 \to G$ is the natural isomorphism and τ: spec $Z_\epsilon \to$ spec F_0 is the finite morphism induced by inclusion. Since dim $X_{(w_1,w_2)} \leq$ rank $T + m(w_1, w_2)$, we conclude that

$$\dim \mathcal{A}' \leq n + 2N - 2 = \dim Z_\epsilon - 2.$$

This inequality is the missing ingredient needed to apply Theorem 3.8 to $\mathcal{O}_\epsilon[G]$, the other hypotheses being confirmed by Proposition 2.7 and Theorem 2.8. We can thus state (incorporating Proposition 3.1)

THEOREM 4.5. *Let $\mathcal{O}_\epsilon[G]$ be as stated in* (2.4), *over the base field $k' = \mathbb{C}$. Then the singular locus of its center Z_ϵ is equal to the non-Azumaya locus.* (*See* (9) *for a geometric description of this locus in terms of the symplectic leaves of G.*) *In particular, the irreducible $\mathcal{O}_\epsilon[G]$-modules of maximal dimension are precisely those whose central characters are non-singular points of* spec Z_ϵ. ▮

4.6. We now consider enveloping algebras of finite dimensional Lie algebras in positive characteristic. In view of (1.10), such enveloping algebras satisfy the hypotheses of Proposition 3.1 and Lemma 3.3, yielding the proposition below. This verifies a conjecture of Panyukov [39, Hypothesis, p. 48], which he proves for the Lie algebra of strictly triangular $n \times n$ matrices by explicit calculation [39, Proposition 3].

PROPOSITION. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field k of positive characteristic, and let $Z = Z(U(\mathfrak{g}))$. Then the locus of points $m \in$* max Z *that occur as Z-annihilators of irreducible $U(\mathfrak{g})$-modules of maximal dimension consists of regular points.* ▮

4.7. In (4.7)–(4.11), $\mathfrak{g}$ will be the Lie algebra of a connected, simply connected, semisimple algebraic group G over an algebraically closed field k of characteristic $p > 0$. Here, p is assumed to be a *good prime* for G; the precise restrictions which this places on p are listed (for the simple components of G) in [17, Sect. 3; 44, Sect. I.4.3], for example. Note that if $p > 5$ then p is good for all G.

Our aim is to consider the question when the hypotheses of Theorem 3.8 are satisfied by the enveloping algebra $U = U(\mathfrak{g})$ of $\mathfrak{g}$. It is well known that U is a Noetherian domain. As seen in Corollary 1.10, U is Auslander regular and Macaulay, of Krull and global dimension equal to $\dim \mathfrak{g}$. Since G is the direct product of its simple normal subgroups $G_1, \dots, G_n$, its Lie algebra $\mathfrak{g}$ is the direct sum of the corresponding Lie algebras $\mathfrak{g}_1, \dots, \mathfrak{g}_n$, so that $U \cong U(\mathfrak{g}_1) \otimes_k \cdots \otimes_k U(\mathfrak{g}_n)$. It is a routine matter to confirm that the conclusion of Theorem 3.8 holds for U if and only if it holds for each $U(\mathfrak{g}_i)$, $i = 1, \dots, n$. We therefore may assume that G is a *simple* algebraic group.

Set $Z = Z(U)$ and $\mathcal{O} = k\langle x^p - x^{[p]} \mid x \in \mathfrak{g} \rangle$. Then $\mathcal{O}$ is a subalgebra of Z isomorphic to the symmetric algebra $S(\mathfrak{g})$, and U is a free $\mathcal{O}$-module of rank $p^{\dim \mathfrak{g}}$ by the PBW Theorem (cf. [17, Sect. 1]). Note that these facts imply that Z is an affine k-algebra and U is a finitely generated Z-module. Set $N = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$; we claim that

$$\text{PI-degree}(U) = p^N. \tag{10}$$

There are two ways to see this. For one, the quotient field $\operatorname{Fract}(Z)$ has dimension $p^{\operatorname{rank} \mathfrak{g}}$ over the quotient field $\operatorname{Fract}(\mathcal{O})$ [25, Lemma 4.2]. On the other hand, since U is a free $\mathcal{O}$-module of rank $p^{\dim \mathfrak{g}}$, the quotient division ring $\operatorname{Fract}(U)$ must have dimension $p^{\dim \mathfrak{g}}$ over $\operatorname{Fract}(\mathcal{O})$. Hence, $\operatorname{Fract}(U)$ has dimension p^{2N} over $\operatorname{Fract}(Z)$. Since $\operatorname{Fract}(Z)$ equals the center of $\operatorname{Fract}(U)$ (e.g., [36, Theorem 13.6.5]), (10) follows. Alternately, irreducible U-modules corresponding to any character χ of $\mathcal{O}$ have dimension at most p^N [17, Proposition 1.5], and this dimension is achieved for "regular semisimple" characters [17, Corollary 3.6]. Thus, (10) follows by Proposition 3.1(a).

We recall some (standard) ideas from [17, Sect. 1; 41, Sect. 3.4]. Let V be an irreducible U-module, so that $\mathcal{O}$ operates by scalars on V; thus, there is a character $\chi^V\colon \mathcal{O} \to k$ such that $zv = \chi^V(z)v$ for $z \in \mathcal{O}$ and $v \in V$. For $x \in \mathfrak{g}$ and $v \in V$ we may write

$$(x^p - x^{[p]})v = \chi_V(x)v,$$

where $\chi_V\colon \mathfrak{g} \to k$ is the composition of χ^V with the map $x \mapsto x^p - x^{[p]}$. Since the pth power map is bijective on k, we can define $\lambda_V\colon \mathfrak{g} \to k$ by the rule

$$\lambda_V(x) = \chi_V(x)^{1/p} = \chi^V(x^p - x^{[p]})^{1/p}.$$

It is easy to check that $\lambda_V \in \mathfrak{g}^*$ (see, e.g., [41, Sect. 3.4]); λ_V is called the *p-character* of V. Conversely, each element $\lambda \in \mathfrak{g}^*$ arises from a character of $\mathcal{O}$ just as above. Namely, since the map $x \mapsto x^p - x^{[p]}$ is injective, and since $\mathcal{O}$ is a symmetric algebra, λ induces a character χ on $\mathcal{O}$ such that $\chi(x^p - x^{[p]}) = \lambda(x)^p$ for $x \in \mathfrak{g}$.

Let $\lambda \in \mathfrak{g}^*$ and χ the corresponding character of $\mathcal{O}$, as above. Set $m_\lambda = \ker \chi$, a maximal ideal of $\mathcal{O}$. Since $U/m_\lambda U$ is a finite dimensional k-algebra (of dimension $p^{\dim \mathfrak{g}}$), there are (up to isomorphism) only finitely many irreducible U-modules with p-character λ. The group G acts on $\mathfrak{g}^*$ by the coadjoint action, and if λ, λ' are p-characters in the same G-orbit then $U/m_\lambda U \cong U/m_{\lambda'} U$. In the light of (10) and Proposition 3.1, we must address the question: For which p-characters λ do all the irreducible U-modules corresponding to λ have dimension p^N?

Recall that an element x of $\mathfrak{g}^*$ (or of $\mathfrak{g}$) is called *regular* if the centralizer $C_G(x)$ has dimension equal to $\operatorname{rank} G$ under the coadjoint (resp., the adjoint) action (e.g., [51, Definition 4.2]). Here $C_G(x) = \{g \in G \mid (\operatorname{Ad} g)x = x\}$. It is known that rank G is the minimum possible dimension for such centralizers (cf. [51, Lemma 4.1]).

PROPOSITION 4.8. *Retain the notation and hypotheses of* (4.7) (*with G simple as an algebraic group*). *Assume that p does not divide $r+1$ in case G is of type A_r.*

(i) [51, *Theorem* 4.12] *The regular elements of $\mathfrak{g}$ form an open subset of $\mathfrak{g}$ with complement of codimension* 3.

(ii) *The locus of non-Azumaya points of Z has codimension at least* 3 *in Z.*

Proof. By [44, Lemma I.5.3] or [47, Sect. 3.6, Proposition 1], $\mathfrak{g}$ admits a non-degenerate G-invariant trace form.

(ii) The existence of the trace form provides a G-equivariant isomorphism of $\mathfrak{g}$ with $\mathfrak{g}^*$, so that (i) holds with $\mathfrak{g}^*$ in place of $\mathfrak{g}$. By Premet's theorem [41, Corollary 3.11], if V is an irreducible U-module whose p-character λ is regular, then

$$p^N \mid \dim V. \tag{11}$$

In view of (10) and Proposition 3.1, (11) can be improved to read $\dim V = p^N$. That is, every point of max Z lying over m_λ is in the Azumaya locus of U.

As mentioned in (4.7), there is a natural isomorphism of $\mathcal{O}$ with $S(\mathfrak{g})$, which we identify with the coordinate ring of the affine variety $\mathfrak{g}^*$. Veldkamp's theorem thus says that the set

$$X := \{m_\lambda \mid \lambda \in \mathfrak{g}^*;\ \lambda \text{ regular}\}$$

is an open subset of spec $\mathcal{O}$ with complement of codimension 3. Since Z is module-finite over $\mathcal{O}$, the restriction map $\rho\colon \operatorname{spec} Z \to \operatorname{spec} \mathcal{O}$ has finite fibres; therefore $\rho^{-1}(X)$ is an open subset of spec Z with complement of codimension 3. By the previous paragraph, $\rho^{-1}(X)$ is contained in the Azumaya locus of U, and (ii) follows. ∎

4.9. Proposition 4.8(ii) can fail in case G is of type A_r with p dividing $r+1$, as we saw in Example 3.4(ii). Namely, if $\mathfrak{g} = \mathfrak{sl}_2(k)$ where k is a field of characteristic 2, then the non-Azumaya locus of $U(\mathfrak{g})$ has codimension 1.

4.10. The observations in (4.7) can now be combined with Proposition 4.8 to enable us to state the following consequence of Theorem 3.8 and Proposition 3.1, partially answering a question of Premet [41, Sect. 4.4, Question 2].

THEOREM. *Let $\mathfrak{g}$ be the Lie algebra of a connected, simply connected, semisimple algebraic group G over an algebraically closed field k of characteristic $p > 0$. Let U be the enveloping algebra of $\mathfrak{g}$. Assume that*

(i) *p is a good prime for G;*

(ii) *G has no component of type A_r with $r \equiv -1 \pmod p$.*

Then the locus of non-Azumaya points of the center $Z(U)$ is equal to the singular locus of $Z(U)$. In particular, the irreducible representations of $\mathfrak{g}$ of maximal dimension are precisely those whose central characters are non-singular points of spec $Z(U)$. ∎

4.11. The second conclusion of Theorem 4.10 was proved for $\mathfrak{g} = \mathfrak{sl}_n(k)$ by Panov [38, Corollary of Theorem 5].

Hypothesis (ii) of Theorem 4.10 cannot be omitted, as is demonstrated by the algebra $\mathfrak{sl}_2(k)$ in characteristic 2 (Example 3.4(ii)). In fact, it seems reasonable to conjecture that the conclusion of Theorem 4.10 fails to hold whenever $\mathfrak{g}$ has some $\mathfrak{sl}_{np}(k)$ as a summand.

4.12. Notwithstanding Examples 3.4 and the remark in (4.11), there are other non-abelian modular Lie algebras besides those listed in Theorem 4.10 for which the conclusion of the latter theorem holds. For example, Premet has pointed out to us a similar analysis that applies to the *Witt algebra* $W(1)$; we thank him for permission to sketch the argument here. Thus let

$$\mathfrak{g} = W(1) := \operatorname{Der}(k[x]/(x^p)),$$

where k is an algebraically closed field of characteristic $p > 3$. Then $\mathfrak{g}$ is spanned by the elements $e_i = x^{i+1}(d/dx)$ for $i = -1, \dots, p-2$, and

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j} & \text{if } -1 \le i+j \le p-2 \\ 0 & \text{otherwise.} \end{cases}$$

In 1941, Chang [8] described all the irreducible representations of $\mathfrak{g}$ (see [48] for a short proof of Chang's result); in particular, the maximum dimension is $p^{(p-1)/2}$ (see [48, p. 601]).

As in (4.7), set $Z = Z(U(\mathfrak{g}))$ and $\mathcal{O} = k\langle x^p - x^{[p]} \mid x \in \mathfrak{g}\rangle$, and for $\lambda \in \mathfrak{g}^*$ let m_λ denote the kernel of the corresponding character of $\mathcal{O}$ (that is, the character χ such that $\chi(x^p - x^{[p]}) = \lambda(x)^p$ for $x \in \mathfrak{g}$). Now set

$$X := \{m_\lambda \mid \lambda \in \mathfrak{g}^* \text{ with } \lambda(e_{p-3}) \neq 0 \text{ or } \lambda(e_{p-2}) \neq 0\},$$

an open subset of spec $\mathcal{O}$ with complement of codimension 2. By Chang's work (cf. [48]), the inverse image of X under the restriction map ρ: spec $Z \to$ spec $\mathcal{O}$ consists of Z-annihilators of irreducible $U(\mathfrak{g})$-modules of dimension $p^{(p-1)/2}$, that is, $\rho^{-1}(X)$ is contained in the Azumaya locus of $U(\mathfrak{g})$.

The above observations allow us to give the following final application of Theorem 3.8 and Proposition 3.1:

THEOREM. *Let $\mathfrak{g}$ be the Witt algebra $W(1)$ over an algebraically closed field k of characteristic p > 3, and let U be the enveloping algebra of $\mathfrak{g}$. Then the irreducible representations of $\mathfrak{g}$ of maximal dimension are precisely those whose central characters are non-singular points of* spec $Z(U)$. ∎

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